What you'll Learn About
How to find the error of a series that does not alternate

Lagrange Error Bound/Taylors Inequality/Remainder Estimation Theorem

1. Give the first term of the series for $f(x) = e^x$ centered at $x = 0$

   1

2. Find the approximation for P(.1) = 1

3. Find f(.1) $= e^{.1} = 1.101570918$

4. How accurate is the approximation. $|e^{.1} - 1| = .105170918$

5. What is the value of the next term of the polynomial at x = .1

   next term = X → .1

1. Give the first two terms of the series for $f(x) = e^x$ centered at $x = 0$

2. Find the approximation for P(.1)

3. Find f(.1)

4. How accurate is the approximation. $|e^{.1} - P_2(x)| = .00570918$

5. What is the value of the next term of the polynomial at x = .1

   .005

1. Give the first three terms of the series for $f(x) = e^x$ centered at $x = 0$

2. Find the approximation for P(.1)

3. Find f(.1)

4. How accurate is the approximation. $|e^{.1} - P_3(x)| = .000170918$

5. What is the value of the next term of the polynomial at x = .1 $.000\overline{16}$

~~*1. Give the first 4 terms of the series for $f(x) = e^x$ centered at $x = 0$*~~

1) 4 Terms e^x centered at $x=0$

$1 + x + \frac{x^2}{2!} + \frac{x^3}{3!}$

.00000460487

$|f(x)-P(x)| \le R$

Where

$R = \dfrac{\left|\begin{pmatrix}\text{Max of the} \\ \text{next derivative} \\ \text{on the given} \\ \text{interval}\end{pmatrix}(x-c)^{n+1}\right|}{(n+1)!}$

Where **x-c** is the distance from the center

Where **n** is the order

We must build the next term a little bit bigger to have a good boundary for the error.

2. *Use Taylors Inequality to determine the error bound* $|f(x)-P(x)| \le R$ from $0 \le x \le .1$

$f(x)=e^x$

$f'(x)=e^x$

$f''(x)=e^x$

$f'''(x)=e^x$

$f^4(x)=e^x$

$f^4(0)=e^0=1$

$x=0$

$\frac{f^4(0)x^4}{4!}$

$\frac{1x^4}{4!}$

$x=.1$

$\frac{f^4(.1)x^4}{4!}$

$\boxed{\frac{e^{.1}x^4}{4!}}$

Formula

$\boxed{R=\frac{e^{.1}(.1)^4}{4!}}$

1. *Find the 3rd order polynomial of the series for* $f(x)=\frac{1}{(1-x)^2}$ *centered at* $x=0$

$f(x)=(1-x)^{-2}$

$f'(x)=+2(1-x)^{-3}$

$f''(x)=+6(1-x)^{-4}$

$f'''(x)=+24(1-x)^{-5}$

$f(0)=1$

$f'(0)=2$

$f''(0)=6$

$f'''(0)=24$

$P_3(x)=1+2x+\frac{6x^2}{2}+\frac{24x^3}{3!}$

Remember, whenever you see this, $|f(x)-P(x)| \le R$, *you are finding error bound*

whenever you see this, $|f(x)-P(x)|$, *you are finding the actual error between the function and the approximation from the polynomial*

2. *Find the Lagrange error bound* $|f(x)-P(x)| \le R$ *for the series between* $0 \le x \le .2$

Build Formula

$f^4(x)=120(1-x)^{-6}$

$=\frac{120}{(1-x)^6}$

$x=0$

$\frac{f^4(0)x^4}{4!}$

$\frac{120x^4}{4!}$

$x=.2$

$\frac{f^4(.2)x^4}{4!}$

$\boxed{\frac{\frac{120}{(.8)^6}x^4}{4!}}$

The distance from center

$\boxed{\frac{\frac{120}{(.8)^6}(.2)^4}{4!}=R}$

4) Write the 2nd order Taylor Polynomial for f(x) = cosx at x = $\frac{\pi}{4}$.
Then use Taylors Inequality to determine the error bound at x = 42°

$f(x) = \cos x \qquad f\left(\frac{\pi}{4}\right) = \sqrt{2}/2$

$f'(x) = -\sin x \qquad f'(\pi/4) = -\sqrt{2}/2$

$f''(x) = -\cos x \qquad f''(\pi/4) = -\sqrt{2}/2$

$f'''(x) = \sin x$

$\frac{42\pi}{180} \le x \le \frac{\pi}{4}$

$R = \left| \frac{\frac{\sqrt{2}}{2}\left(\frac{42\pi}{180} - \frac{\pi}{4}\right)^3}{3!} \right|$

$P_2\left(x - \frac{\pi}{4}\right) = \frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}\left(x - \frac{\pi}{4}\right) - \frac{\frac{\sqrt{2}}{2}\left(x - \frac{\pi}{4}\right)^2}{2!}$

Error Bound Formula

$x = 42\pi/180$

$\frac{\sin 42\pi/180 \left(x - \frac{\pi}{4}\right)^3}{3!}$

$x = \pi/4$

$\boxed{\frac{\sin \pi/4 \left(x - \frac{\pi}{4}\right)^3}{3!}}$

5) Write the 1st degree Taylor Polynomial for f(x) = arcsinx at x = 0.
Then use Taylors Inequality to determine the error bound at x = .2

$f(x) = \arcsin x \qquad f(0) = 0$

$(1 - x^2)^{-1/2} = f'(x) = \frac{1}{\sqrt{1 - x^2}} \qquad f'(0) = 1$

$R = \left| \frac{\frac{.2}{(1 - .2^2)^{3/2}} (.2)^2}{2!} \right|$

$P_1(x) = x$

Error Bound Formulaa

at $x = 0$

$\frac{0x^2}{2!}$

$x = .2$

$\boxed{\frac{\frac{.2}{(1 - .2^2)^{3/2}} x^2}{2!}}$

$f''(x) = -\frac{1}{2}(1 - x^2)^{-3/2} \cdot -2x$

$f''(x) = \frac{x}{(1 - x^2)^{3/2}}$

	Summary of Error Bound For an <u>Alternating Series</u> – Use the next term For a series that is <u>Not Alternating</u> 1. Write down the formula for the next derivative. 2. Find the value of the next derivative at the ends of the interval and the center. 3. Whichever value is bigger is the value you use to build your error bound term

$f(x)=\sum_{n=0}^{\infty} c_n x^n$	INTERVAL OF CONVERGENCE	RADIUS OF CONVERGENCE
$\frac{1}{1-x}=\sum_{n=0}^{\infty} x^n = 1+x+x^2+x^3+\cdots$	$(-1,1)$	1
$e^x=\sum_{n=0}^{\infty}\frac{x^n}{n!}=1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\cdots$	$(-\infty,\infty)$	∞
$\sin x=\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{(2n+1)!}$ $=x-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}+\cdots$	$(-\infty,\infty)$	∞
$\cos x=\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n}}{(2n)!}$ $=1-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}+\cdots$	$(-\infty,\infty)$	∞
$\tan^{-1} x=\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{2n+1}$ $=x-\frac{x^3}{3}+\frac{x^5}{5}-\frac{x^7}{7}+\cdots$	$(-1,1]$	1
$\ln(1+x)=\sum_{n=1}^{\infty}(-1)^{n-1}\frac{x^n}{n}$ $=x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\cdots$	$(-1,1]$	1